Disordered ensembles of random matrices

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It is shown that the families of generalized matrix ensembles recently considered which give rise to an orthogonal invariant stable Lévy ensemble can be generated by the simple procedure of dividing Gaussian matrices by a random variable. The nonergodicity of this kind of disordered ensembles is investigated. It is shown that the same procedure applied to random graphs gives rise to a family that interpolates between the Erdös-Renyi and the scale free models.

I. INTRODUCTION

The classes of random matrix ensembles introduced by Wigner in the 1950s have found great sucess partly after being connected with quantum manifestations of chaos in physical systems [1]. In turn, this success generated great activity and extensions and generalizations of those ensembles have occurred. In obtaining the Gaussian ensembles, Wigner adapted the Wishart ensembles well known to statistitians. Some of the extensions of the Gaussian ensembles can also be considered as applications of known processes in statistics. For instance, models to describe symmetry breaking have been constructed by adding two random matrices, one block diagonal and the other its complement [2]. Here we consider a random process in which a new random quantity is generated by taking not the sum but the ratio or the product of two other independent ones.

In a previous paper [3], an alternative to Shannon information entropy, namely Tsallis-Renyi information [4] was used to introduce a new family of generalized matrix ensembles (see also Ref. [5]). One of the main features of this ensemble is the power-law characteristic of its statistical properties. In particular, it was shown that individual matrix elements behave similar to the elements of the so-called Lévy matrices [6] (after the publication of Ref. [3], Klauder and Muttalib obtained an even more general family [7] on similar lines). One of the purposes of this note is to show that all these families can be obtained, in fact, by the simple procedure presented in the next section.

II. THE MODEL

Let $H_G(\alpha)$ be a random matrix of dimension N and variance $1/2\alpha^2$ and let its probability distribution be

$$P_G(H;\alpha) = \left(\frac{\beta\alpha}{\pi}\right)^{f/2} \exp(-\alpha\beta \mathrm{tr}H^2),$$

$$(1)$$

The matrices of the Gaussian ensemble are specified by α. In Eq. (1), f is the number of independent matrix elements $f=N+\beta N(N-1)/2$ and β is the Dyson index $\beta=1,2,4$ for GOE, GUE, and GSE (here and in what follows the subindex G indicates Gaussian). The distribution is normalized with respect to the measure $dH=\Pi_1^N dH_{ii}\Pi_{j>i}\Pi_{k=1}^\beta \sqrt{2}dH_{ij}^k$.

Take now a positive random variable ξ with a normalized density probability distribution $w(\xi)$ with average $\bar{\xi}$ and variance σ_ξ^2 and introduce a new matrix ensemble by the following relation (product of random variables has been considered in the context of covariace matrices [8])

$$H(\alpha,\xi) = \frac{H_G(\alpha)}{\sqrt{\xi/\bar{\xi}}}.$$

$$(2)$$

In this way, an external source of randomness is superimposed to the fluctuations of the Gaussian matrix $H_G(\alpha)$. A random process in which there is a competition between two types of random variables is typical of disordered systems or, in the case of Ising models, spin glasses [9]. As the two types of randomness are independent, one can be kept frozen, quenched in technical terms, while the fluctuations of the other continue to operate. Here the disorder is represented by ξ which is the quenched variable in opposition to the randomness of the Gaussian matrices. We may refer to Eq. (2) as a disordered ensemble.

From Eq. (2), we deduce that the joint distribution of a set of $n \leq f$ matrix elements is given by

$$p(h_1, h_2, \ldots, h_n; \alpha) = \left(\frac{\beta\alpha}{\pi\bar{\xi}}\right)^{n/2} \int d\xi w(\xi)\xi^{n/2}$$

$$\times \exp\left(-\frac{\beta\alpha\xi}{\bar{\xi}}\sum_{i=1}^n h_i^2\right),$$

$$(3)$$

where $h_i=H_{ij}$ for the diagonal and $h_i=\sqrt{2}H_{ij}$ for the off-diagonal elements. Equation (3) shows that matrix elements are correlated. As a particular case, for $n=f$, Eq. (3) leads to the ensemble distribution

$$P(H;\alpha) = \int d\xi w(\xi)\left(\frac{\beta\alpha\xi}{\pi\bar{\xi}}\right)^{f/2} \exp\left(-\frac{\beta\alpha\xi}{\bar{\xi}}\mathrm{tr}H^2\right), \quad (4)$$

where the term after $w(\xi)$ is just Eq. (1) with α replaced by $\alpha\xi/\bar{\xi}$. Expressions such as Eq. (4) are being considered as instances of superstatistics [10].

The relation (2) makes straightforward to do numerical simulations in terms of Gaussian matrices. However, it may also be useful to directly generate matrices of the ensemble

(taking into account the corrrelations among their elements). This can be done through the identity

$$p(h_1, \ldots, h_f) = p(h_1) \prod_{n=2}^{f} \frac{p(h_1, \ldots, h_n)}{p(h_1, \ldots, h_{n-1})}, \quad (5)$$

where each fraction gives the conditional probability for the nth element once the $n-1$ previous ones are given. This equation provides a way to sequentially generate all the matrix elements. At each step, a new element, say the nth, is sorted using Eq. (3) that implies

$$h_n = \frac{h_G(\alpha)}{\sqrt{\xi_n/\bar{\xi}}}, \quad (6)$$

where h_G is a Gaussian variable and ξ_n is another random variable sorted from the distribution

$$w_n(\xi) = w(\xi)\xi^{(n-1)/2} \exp\left(-\frac{\beta\alpha\xi}{\bar{\xi}}\sum_{i=1}^{n-1} h_i^2\right) \Bigg/ \int d\xi w(\xi)\xi^{(n-1)/2} \exp\left(-\frac{\beta\alpha\xi}{\bar{\xi}}\sum_{i=1}^{n-1} h_i^2\right), \quad (7)$$

which is univariate since all the previous $n-1$ elements have already been determined.

By generating matrices fixing, in the process, a set of values $\xi_1, \xi_2, \ldots, \xi_f$ we are, in the language of the disordered systems, quenching the disorder. The differences among matrices generated with different sets of ξ depend on the width of the distribution $w(\xi)$ and one can expect that for wide $w(\xi)$ due to the large spread among the matrices averages performed running along one spectrum will not coincide with averages over the ensemble of matrices. This kind of behavior characterizes nonergodicity.

Turning now to eigenvalues and eigenvectors, we observe that we have an ensemble invariant under unitary transformation in which, as it occurs with the Gaussian ensembles, the joint distribution of eigenvalues and eigenvector factorizes. The eigenvectors behave as those of the Gaussian ensembles and we can integrate them out to obtain for the eigenvalues the joint distribution

$$P(E_1, \ldots, E_N; \alpha) = \int d\xi w(\xi)(\alpha\xi/\bar{\xi})^{N/2} P_G\left(x_1, \ldots, x_N; \frac{\beta}{2}\right), \quad (8)$$

where $x_i = \sqrt{\alpha\xi/\bar{\xi}}E_i$ and

$$P_G\left(x_1, \ldots, x_N; \frac{\beta}{2}\right) = K_N^{-1} \exp\left(-\frac{\beta}{2}\sum_{k=1}^{N} x_k^2\right) \prod_{j>i} |x_j - x_i|^\beta, \quad (9)$$

with K_N being a normalization constant.

From Eq. (8), measures of the generalized family can be calculated by weighting the corresponding measures of the Gaussian ensembles with the $w(\xi)$ distribution. Integrating,

for instance, Eq. (8) over all eigenvalues but one and multiplying by N, the eigenvalue density is expressed in terms of the Wigner's semicircle law [11] as

$$\rho(E; \alpha) = \frac{\sqrt{2\alpha}}{\pi} \int d\xi w(\xi)(\xi/\bar{\xi})^{1/2}\sqrt{2N - 2\alpha\xi E^2/\bar{\xi}}, \quad (10)$$

where the condition $\alpha\xi E^2 < N$ on ξ has to be satisfied.

As previously stated, the introduction of the disorder represented by the variable ξ, breaks in principle the ergodicity of the Gaussian ensembles. Let $N(L) = \int_{E-L/2}^{E+L/2} dE' \rho(E')$ be the average number of eigenvalues in the interval $[E-L/2, E+L/2]$ for an ensemble with eigenvalue density $\rho(E)$. The variance $\Sigma^2(L)$ of the number of eigenvalues in that interval can be expressed in terms of the two-point correlation function $R(E_1, E_2)$ by

$$\Sigma^2(L) = \int_{E-L/2}^{E+L/2} dE_1 \int_{E-L/2}^{E+L/2} dE_2 R(E_1, E_2) + N(L) - N^2(L). \quad (11)$$

By using in the above integrals "unfolded" variables $y_{1,2} = \int^{E_{1,2}} \rho(v)dv$ and assuming stationarity along the spectrum (11) becomes the usual expression

$$\Sigma^2[N(L)] = N(L) - 2\int_0^{N(L)} dr[N(L) - r]Y(r) \quad (12)$$

for the number variance in terms of the two-point cluster function $Y_2(r)$ [11]. Ergodicity implies [12] the vanishing of

$$\text{var}\rho = [\rho(E)]^2 \Sigma^2[N(L)]/N^2(L) \quad (13)$$

when $L \to \infty$. For the disordered ensemble we have

$$\Sigma^2(N) = \int d\xi w(\xi)[\Sigma_G^2(N_G) - N_G + N_G^2] + N - N^2, \quad (14)$$

where Σ_G^2 is given by Eq. (12). In Eq. (14), N_G is the average number of eigenvalues in the interval L calculated with a semicircle density dependent on ξ; on the other hand, N is the average number of levels calculated with the ensemble density (10). It is easy to see that, in Eq. (14), the linear term in N is cancelled while the N_G^2 averaged with the distribution $w(\xi)$ deviates from N^2. If the difference between these two terms increases for large $N(L)$ with a power equal or greater than two, then the variance of the density fluctuations given by Eq. (13) does not asymptotically vanish and nonergodicity follows.

Consider now a particular choice of the distribution $w(\xi)$. Note that the factor multiplying the Gaussian matrices in Eq. (2) acts on the variance of the Gaussian ensembles. In order to investigate ensembles showing heavy-tailed densities it is convenient to choose $w(\xi)$ to be the gamma distribution

$$w(\xi) = \exp(-\xi)\xi^{\bar{\xi}-1}/\Gamma(\bar{\xi}) \quad (15)$$

that becomes a χ^2 distribution for integer $2\bar{\xi}$. From Eq. (15) $\sigma_\xi = \sqrt{\bar{\xi}}$, showing that $\bar{\xi}$ controls the behavior of the distribution $w(\xi)$. It becomes more localized when $\bar{\xi}$ increases and we should then expect to recover the Gaussian ensembles.
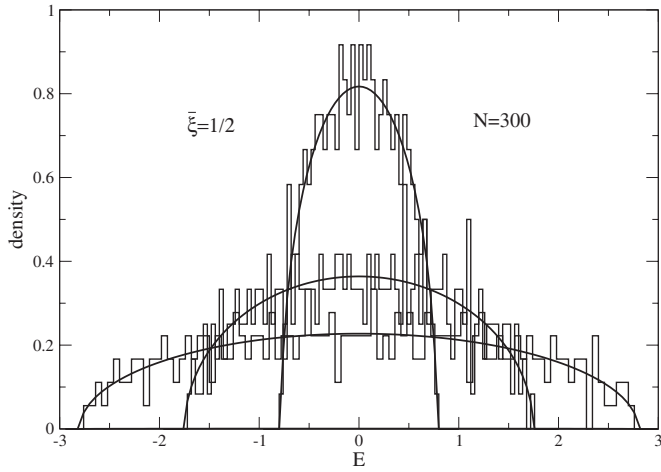
FIG. 1. Histogram: the eigenvalue density of three matrices of size $N=300$ generated using Eqs. (6) and (15) with $\bar{\xi}=1/2$; full lines: semicircle densities that reproduce the variance of the eigenvalues of each one of the three matrices.

However, for smaller values of $\bar{\xi}$, departures from the Gaussian case will be observed. Indeed, by substituting Eq. (15) in Eq. (4) we find

$$P(H;\alpha,\bar{\xi}) = \left(\frac{\beta\alpha}{\pi\bar{\xi}}\right)^{f/2} \frac{\Gamma\left(\frac{1}{q-1}\right)}{\Gamma(\bar{\xi})} \left(1 + \frac{\beta\alpha}{\bar{\xi}}\mathrm{tr}H^2\right)^{1/(1-q)} \tag{16}$$

for the ensemble density distribution, where

$$\frac{1}{q-1} = \bar{\xi} + \frac{f}{2}, \quad \text{with } q > 1. \tag{17}$$

Equation (16) is just Eq. (4) of Ref. [3]. In Ref. [3] it was derived using a generalized maximum entropy principle [4] with q being identified with the Tsallis entropic parameter.

Substituting Eq. (15) in Eq. (3) for $n=1$ [13]

$$p(h;\alpha,\bar{\xi}) = \left(\frac{\beta\alpha}{\pi\bar{\xi}}\right)^{1/2} \frac{\Gamma(\bar{\xi}+1/2)}{\Gamma(\bar{\xi})} \left(1 + \frac{\beta\alpha}{\bar{\xi}}h^2\right)^{-\bar{\xi}-1/2} \tag{18}$$

for the density distribution of a given matrix element. Since for large $|h|$, $p_\beta(h;\alpha,\bar{\xi}) \sim 1/|h|^{2\bar{\xi}+1}$, Eq. (18) exhibits the power-law character of the distribution. It is important to remark that, apart from the lack of independence, the marginal distribution of the matrix elements have the same kind of distribution, namely one with an asymptotic power-law behavior, as the i.i.d. ones of the ensemble of Lévy matrices [6].

In Fig. 1 the eigenvalue density for three realizations of the ensemble generated using the above random process with $\bar{\xi}=1/2$ is histogrammed and compared with the semicircle law. We recall that for $\bar{\xi}=1/2$ the matrix elements are Cauchy, $\frac{1}{\pi}\frac{1}{1+x^2}$, distributed [see Eq. (18)]. It is seen that the individual matrices of large sizes are Gaussian ensemble matrices as they should. As a comparison, in Fig. 2, it is shown
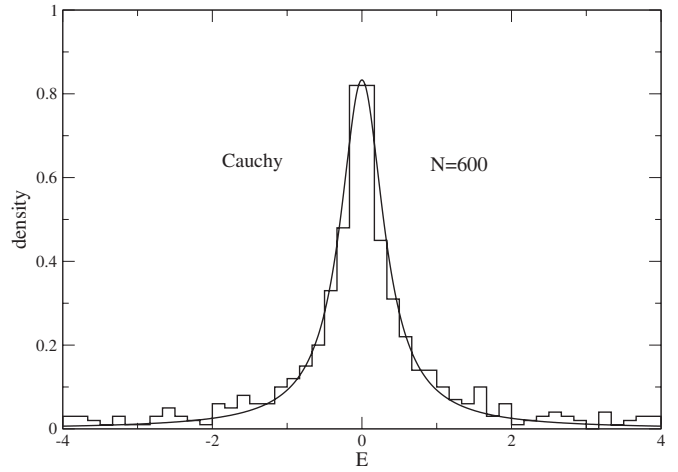

FIG. 2. The eigenvalue density of one Lévy matrix of size $N=600$ whose elements are Cauchy distributed compared to a Cauchy distribution.

the eigenvalue density of just one Lévy matrix of large size whose matrix elements also follow the Cauchy distribution. We can see that although individual matrix elements of the two ensembles are identically distributed, their eigenvalue density behaves in a completely different way. While individual Lévy matrices of large sizes do not depart from the ensemble average, matrices generated according to Eq. (6) show large fluctuations.

Of course, the result shown in Fig. 1 indicates strong nonergodicity. This is confirmed by the ensemble number variances shown in Fig. 3. The parabolic behavior seems to persist even for large values of the parameter $\bar{\xi}$, showing that the ensemble is nonergodic.

III. RANDOM GRAPHS

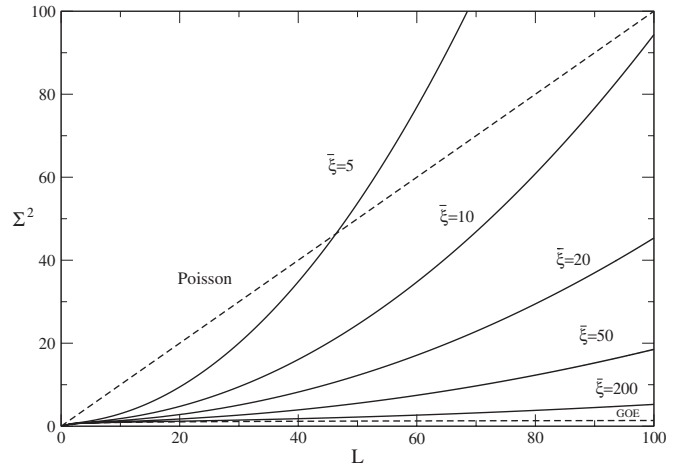Other systems in which nonergodicity may play an important role are networks and their associated graphs. We now

FIG. 3. Full lines: the number variances calculated with Eq. (14) for the values $\bar{\xi}=5$, 10, 20, 50, and 200 as indicated in the figure; dashed lines: the linear Poisson number variance and the GOE number variance.

show how the present approach can be applied in random graph theory [14]. A graph is an array of points (nodes) connected by edges. It is completely defined by its adjacency matrix A whose elements A_{ij} have value 1(0) if the pair (ij) of nodes is connected (disconnected). The diagonal elements are taken equal to zero, i.e., $A_{ii}=0$. Adjacency matrices of graphs in which the connections are randomly set, are real symmetric random matrices. The classical random graph model proposed by Erdös-Renyi (ER) is simply defined by giving a fixed probability p that a given pair of nodes is connected, independently of the others [15].

We start by showing that the ER model can be considered as the equivalent in random graph theory to the Wigner model of Gaussian matrices. In fact, the joint matrix element distribution of its adjacency matrix A can be written as

$$P_{\text{ER}}(A,\alpha) = [1 + \exp(-\alpha)]^{-f} \exp\left(-\frac{\alpha}{2}\text{tr}A^2\right), \quad (19)$$

where $f = \frac{N(N-1)}{2}$ with N, the size of matrix, being equal to the number of nodes. Equation (19) is just the defining equation (1) of the GOE ($\beta=1$) ensemble with the constraint that the matrix elements can only take the values 0 and 1 imposed by the measure

$$dH = \prod_{1}^{N} dH_{ii}\delta(H_{ii})\prod_{j>i} \sqrt{2}dH_{ij}[\delta(H_{ij}) + \delta(1 - H_{ij})]. \quad (20)$$

From Eq. (19) it follows that the marginal distribution of a given matrix element, say A_{ij}, is

$$P_{\text{ER}}(A_{ij},\alpha) = \frac{\exp(-\alpha A_{ij})}{1 + \exp(-\alpha)} = \begin{cases} \dfrac{\exp(-\alpha)}{1 + \exp(-\alpha)}, & \text{if } A_{ij} = 1, \\[3mm] \dfrac{1}{1 + \exp(-\alpha)}, & \text{if } A_{ij} = 0, \end{cases} \quad (21)$$

which means that the probability p that defines the ER model is connected to the parameter α by the relation

$$\alpha = \ln\left(\frac{1}{p} - 1\right). \quad (22)$$

Since the probability p is defined in the interval [0,1], the domain of variation of α is $]\infty,-\infty[$. This suggests that the statistical properties of the ER model must show a symmetry with respect to the point $\alpha=0$ (or $p=1/2$).

It is important to remark that although Eq. (19) has the same structure as Eq. (1) there are striking differences between the two models. Despite the presence of the trace in Eq. (19), the discrete nature of matrix elements imposed by the measure, (20), destroys the rotational invariance and prevents the factorization of the joint distribution of eigenvalues and eigenvectors. The parameter α is just a scaling parameter in the Gaussian case. In contrast, the properties of ER model depend strongly on the value of the probability p, and here α plays an essential role. Notice also that, contrarily to the Gaussian cases, the adjacency matrices form an ensemble

with a finite number of matrices. It is convenient in the study of the graphs, to introduce the scaling $p \sim N^{-z}$ ($z>0$). For instance, connectivity properties of the graph are characterized by z.

An analytical expression of the spectral density for arbitrary values of the probability p and matrix size N is an unsolved problem [16]. However, when p is fixed and N is very large, the density can be deduced in the following way. A is a symmetric non-negative matrix with maximum principal eigenvalue E_1 its value is close to the nonzero eigenvalue of the constant matrix $\langle A \rangle$ with elements equal to the average of the A elements, i.e., $\langle A \rangle_{ij}=p$. As the only nonzero eigenvalue of a constant matrix is equal to the product of its size by the element, we conclude that $E_1=pN$. Because of this linear dependence with N, for fixed p the largest eigenvalue grows faster than the others as the matrix size increases. In this case, for very large matrices the other eigenvalues have asymptotically the same eigenvalue density of the eigenvalues of the matrix $A-\langle A \rangle$. This density can be obtained from the moments of the trace of the powers of the matrix and one finds that it obeys the Wigner semicircle law [14]

$$\rho_{\text{ER}}(E,\alpha) = \begin{cases} \dfrac{1}{2\pi\sigma^2}\sqrt{4N\sigma^2 - E^2}, & \text{if } |E| < \sqrt{4N\sigma^2}, \\[3mm] 0, & \text{if } |E| > \sqrt{4N\sigma^2}, \end{cases} \quad (23)$$

where σ^2 is the variance of the matrix elements given by

$$\sigma^2 = p(1 - p) = \frac{1}{4\cosh^2(\alpha/2)}. \quad (24)$$

The above argument fails if $p \sim 1/N$ ($z \sim 1$) in which case deviations from the semicircle appear [16,17].

We now introduce a disordered model of random graphs by defining an adjacency matrix with a distribution

$$P(A;\alpha) = \int d\xi w(\xi) \frac{\exp\left(-\dfrac{\alpha\xi}{2}\text{tr}A^2\right)}{[1 + \exp(-\alpha\xi)]^f}. \quad (25)$$

Therefore this generalized model is a superposition of Erdös-Renyi random graphs with distribution $P(A,\alpha\xi)$ weighted with $w(\xi)$ exactly as in Eq. (4) for the disordered Gaussian ensembles. Again the width of the distribution of $w(\xi)$ is a controling parameter and as remarked before the parameter α also plays an essential role. In particular, for $\alpha=0$ the ensemble is just the ER with $p=1/2$.

From Eq. (25) we can derive the probability distribution for a set of matrix elements and use Eq. (5) to define a random process entirely equivalent to the one used to generate matrices of the disordered Gaussian ensemble. As before, a set of probabilities p_n with $n=1,2,3,\ldots,f$ is sequentially generated and, from them, each new matrix element is obtained taking into account those already determined. This means that Eq. (25) defines a model of a disordered correlated graph in which new attachments depend on the ones already existing.

As in the case of the Gaussian ensembles, statistics of the averaged graph (our model) are averages over the ER statistics. For instance, the eigenvalue density is

$$\rho(E;\alpha) = \frac{2}{\pi} \int_0^{\xi_m} d\xi w(\xi) \cosh\left(\frac{\alpha\xi}{2}\right) \sqrt{N - \cosh^2\left(\frac{\alpha\xi}{2}\right)E^2},$$

(26)

where

$$\xi_m = \frac{2}{\alpha} \cosh^{-1}\left(\frac{\sqrt{N}}{E}\right).$$

(27)

We now make for $w(\xi)$ the same choice as before, namely, Eq. (15). As before we expect for large values of $\bar{\xi}$ small fluctuations around ER, whereas for small values they will become large and will govern the asymptotics.

In Fig. 4 we display the density of eigenvalues of the adjacency matrices. When going from z close to 1 to z close to 0, the density goes from a highly picked density with heavy tails towards a Wigner semicircle, showing a crossover which is reminiscent from a scale-free to an ER graph.

IV. CONCLUSIONS

In summary, we have discussed a method to introduce matrix ensembles which preserve unitary invariance presenting distribution with heavy tails. The price to pay to preserve unitary invariance is (i) to abandon the statistical independence of the matrix elements and (ii) to abandon the ergodic property (equivalence of spectral and ensemble averages). There are cases, however, in which only ensemble averages make sense. Consider, for instance, the behavior of individual eigenvalues. Recently, extreme eigenvalues have been a matter of great interest due to the discovery that the distributions they follow, the so-called Tracy-Widom [18] in the case of the Gaussian ensembles, show universality and have
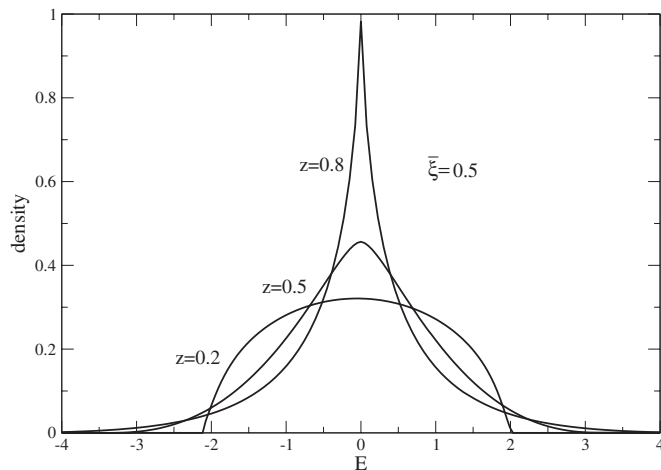

FIG. 4. The eigenvalue density of the disordered random graph model calculated with Eqs. (26) and (15) with $\bar{\xi} = 1/2$ and for values 0.2, 0.5, and 0.8 of the scaling parameter z.

wide applications [19]. The same authors have found growing systems in which an external source induces the extreme values to have a behavior in which there is a competition between their distribution and a Gaussian [20]. In a paper in preparation, we show that the disordered ensemble can be a useful model for this kind of systems.

Let us finally mention that the method discussed here [Eq. (2) with the choice (15) for the probability density function $w(\xi)$] was intended to rederive and to give new insight on models previously studied. By making other choices for $w(\xi)$ new models preserving orthogonal invariance may be introduced (see also Ref. [7]).

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